UDC 532.72

NONSTATIONARY MASS TRANSFER OF A REACTING SPHERICAL PARTICLE IN LAMINAR STREAM OF A VISCOUS FLUID

PMM Vol. 38, №6, 1974, pp. 1041-1047
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(Received February 19, 1974)

The process of the formation of a stationary mass transfer mode for a moving reacting particle is examined. An analytic expression valid for a nonstationary distribution of the concentration of matter in a steady stream of viscous fluid, flowing past a spherical particle, was obtained for the case when at a certain instant a chemical reaction of the first order begins at the surface of the sphere. The problem is solved for small finite Reynolds and Péclet numbers. The solution of the corresponding stationary problem has been obtained in [1]. Paper [2] examined a nonstationary heat transfer of a fluid spherical drop in an inviscid flow with spasmodic change of initial temperature at high Péclet numbers. Paper [3] contains an analysis of the problem of a nonstationary heat transfer of a rigid spherical particle for small Reynolds and Péclet numbers at spasmodic change of temperature of the particle surface. The results obtained in [3] can be used to describe the mass transfer for a moving reacting particle only in the case of a diffusion mode of the chemical reaction.

1. Statement of the problem. A diffusion process is considered for the case of an impenetrable spherical particle, the radius of which is a and around which a stationary stream of a viscous incompressible fluid flows. At large distances from the particle the velocity of the oncoming stream and the concentration of the diffusing material are constant and equal U, c_0, respectively. It is assumed that at an initial instant a chemical reaction of the first order begins at the surface of the particle (e.g. due the particle being warmed up to a certain critical temperature). We choose the spherical system of coordinates r, θ, where the coordinate r is taken with respect to the center of the particle and the angle θ to the direction of the flow velocity at infinity. We represent the equation and the boundary conditions in the form

$$\frac{\partial\xi}{\partial\tau}+v_r\frac{\partial\xi}{\partial r}+\frac{v_\theta}{r}\frac{\partial\xi}{\partial\theta}=\frac{1}{P}\Delta\xi,\quad P=\frac{Ua}{D} \tag{1.1}$$

$$v=\frac{v'}{U},\quad \xi=\frac{c_0-c}{c_0},\quad r=\frac{r'}{a},\quad \tau=\frac{U}{a}t$$

$$\tau=0,\quad r\geqslant 1,\quad \xi=0 \tag{1.2}$$

$$\tau>0,\quad r=1,\quad \frac{\partial\xi}{\partial r}=k(\xi-1),\quad k=k_*a/D \tag{1.3}$$

$$r\to\infty,\quad \xi\to 0 \tag{1.4}$$

Here v_r, v_θ are the velocity components, D is the diffusion coefficient, k_* is the rate of the chemical reaction, the prime denotes dimensional values. We note that the relations (1.1)–(1.4) describe also the process of the heat transfer and for $k=\infty$ correspond to the mode for which the surface temperature is maintained constant. In the problem of the heat transfer it is assumed

$$\xi = \frac{T - T_0}{T_1 - T_0}, \qquad P = \frac{Ua}{\chi}$$

where T_0 is the temperature of the onflowing stream, T_1 is the surface temperature and χ is the coefficient of the thermal diffusivity of the medium. The problem is solved for small finite Péclet and Reynolds numbers. The velocity distribution in the stream is considered to be specified by the expressions obtained in [4]. Introducing the Schmidt number $S = P / R$ we write for the stream function the expressions [4] in the following form:

for the inner region

$$\psi = \psi_* = \frac{1}{4}(r-1)^2(1-\mu^2)\Big[\Big(1 + \frac{3}{8S}P + \frac{9}{40S^2}P^2 \ln P\Big)\Big(2 + \frac{1}{r}\Big) - \frac{3}{8S}P\Big(2 + \frac{1}{r} + \frac{1}{r^2}\Big)\mu\Big]$$

and for the outer region

$$\psi = \frac{\psi^*}{P^2}, \quad \psi^* = \frac{1}{2}\rho^2(1-\mu^2) - \frac{3}{2}SP(1+\mu) \times \Big[1 - \exp\Big(-\frac{\rho}{S}\frac{1-\mu}{2}\Big)\Big] \quad \mu = \cos\theta, \qquad \rho = rP$$

Terms of the order P^2 are not taken into account. The Schmidt number is assumed to be large or equal $O(1)$ (the case of $S \ll 1$ is of small practical interest) and in the solution of the problem is considered constant.

2. Method of solution. We introduce the Laplace transformation for the concentration to be found

$$\zeta = \int_0^\infty e^{-Ps\tau}\xi(r, \mu, \tau)\,d\tau$$

As ξ is limited for $\tau \to \infty$, then ζ is analytic for $\mathrm{Re}\, s > 0$. Taking into account (1.2) the equation obtained from (1.1) for ζ in the inner region has the form

$$\zeta = \zeta_*, \quad P_s{}^2\zeta_* + \frac{P}{r^2}\Big(\frac{\partial\psi_*}{\partial r}\frac{\partial\zeta_*}{\partial\mu} - \frac{\partial\psi_*}{\partial\mu}\frac{\partial\zeta_*}{\partial r}\Big) = \Delta\zeta_* \tag{2.1}$$

and in the outer region

$$\zeta = \zeta^*, \quad s\zeta^* + \frac{1}{\rho^2}\Big(\frac{\partial\psi^*}{\partial\rho}\frac{\partial\zeta^*}{\partial\mu} - \frac{\partial\psi^*}{\partial\mu}\frac{\partial\zeta^*}{\partial\rho}\Big) = \Delta^*\zeta^* \tag{2.2}$$

Here Δ^* is the Laplace operator in the variables ρ, μ. The boundary conditions (1.3), (1.4) yield

$$r = 1, \qquad \frac{\partial\zeta_*}{\partial r} = k\zeta_* - \frac{k}{sP} \tag{2.3}$$

$$\rho \to \infty, \qquad \zeta^* \to 0 \tag{2.4}$$

The problem is solved by the method of matched asymptotic expansions with resepct to the Péclet number. The particular features of the method were examined in [4]. We have to determine ζ in the form

$$\zeta_* = \sum_{n=0}^\infty \alpha_n(P)\,\zeta_n(r, \mu) \quad \text{in the inner region} \tag{2.5}$$

$$\zeta^* = \sum_{n=0}^\infty \alpha^{(n)}(P)\,\zeta^{(n)}(\rho, \mu) \quad \text{in the outer region} \tag{2.6}$$

Moreover, it is required that

$$\frac{\alpha_{n+1}(P)}{\alpha_n(P)} \to 0, \quad \frac{\alpha^{(n+1)}(P)}{\alpha^{(n)}(P)} \to 0, \quad P \to 0$$

3. Construction of the solution. Zero order approximations. The zero order approximation of the outer expansion has the form $\zeta^{(0)}(\rho, \mu) = 0$. For the inner expansion we obtain the Laplace equation $\Delta \zeta_0 = 0$, and we assume $\alpha_0(P) = 1 / P$. For such a choice of α_0 the solution $\alpha_0 \zeta_0$, which satisfies the condition (2.3), represents a Laplace form of zero approximation quoted in [1]

$$\zeta_0 = \frac{q}{sr}, \qquad q = \frac{k}{k+1} \tag{3.1}$$

First approximations. It follows from (3.1) that $\alpha^{(1)}(P) = 1$. The equation for $\zeta^{(1)}(\rho, \mu)$ has the form

$$(\Lambda - s)\zeta^{(1)} = 0, \qquad \Lambda = \Delta^* - \mu\frac{\partial}{\partial\rho} - \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu} \tag{3.2}$$

The solution of Eq. (3.2) vanishing at infinity, is

$$\zeta^{(1)} = \exp\left(\frac{\rho\mu}{2}\right)\left(\frac{\pi}{\rho}\right)^{1/2} \sum_{n=0}^{\infty} A_n K_{n+1/2}\left(\frac{\rho}{2}\sqrt{1+4s}\right) P_n(\mu)$$

Here $K_{n+1/2}(x)$ is MacDonald function, $P_n(\mu)$ is the Legendre polynomial; that branch of the function $\sqrt{1+4s}$ which on the real half-axis $0 \leqslant s < \infty$ yields the arithmetic value of the root, is always taken. The coefficients A_n are determined from the matching conditions with (3.1). Finally we have

$$\zeta^{(1)} = \frac{q}{s\rho}\exp\left[\frac{\rho}{2}\left(\mu - \sqrt{1+4s}\right)\right] \tag{3.3}$$

It follows that $\alpha_1(P) = 1$ and the equation for ζ_1 has the form

$$\Delta\zeta_1 = -\frac{q}{sr^2}\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\mu$$

The solution of this equation satisfying the boundary condition following from (2.3) and the matching condition with (3.3)

$$\zeta_1 = \frac{q\sqrt{1+4s}}{2s}\left(\frac{q}{r} - 1\right) + \frac{q}{s}\left[\frac{1}{2} - \frac{3}{4r} + \frac{3(3-2q)}{8(2-q)}\frac{1}{r^2} - \frac{1}{8r^3}\right]\mu \tag{3.4}$$

Second approximations. In the second approximation $\alpha^{(2)}(P) = P$. We have the equation

$$(\Lambda - s)\zeta^{(2)} = \exp\left(\frac{\rho\mu}{2}\right) L(\rho, \mu) \tag{3.5}$$

$$L(\rho, \mu) = \frac{3}{4}\frac{qS}{s\rho^3}\left(\frac{\sqrt{1+4s}-1}{2S}\rho + \sqrt{1+4s} + \frac{1}{S} + \frac{2}{\rho} + \frac{\sqrt{1+4s}-1}{2S}\mu - \frac{S-1}{S}\mu\right)\exp\left[\frac{\rho}{2}\left(\frac{\mu-1}{S} - \sqrt{1+4s}\right)\right] - \frac{3}{4}\frac{qS}{s\rho^3}\left(\sqrt{1+4s} + \frac{2}{\rho} - \mu\right)\exp\left(-\frac{\rho}{2}\sqrt{1+4s}\right)$$

The solution vanishing at infinity is

$$\zeta^{(2)}(\rho, \mu) = q\rho^{-1/2}\exp\left(\frac{\rho\mu}{2}\right)\sum_{n=0}^{\infty}\eta_n(\rho)P_n(\mu) \tag{3.6}$$

$$\eta_n = K_{n+1/2}\left(\frac{\rho}{2}\sqrt{1+4s}\right)\int_{\rho}^{C_n} I_{n+1/2}\left(\frac{\rho}{2}\sqrt{1+4s}\right)L_n(\rho)\,d\rho -$$

$$I_{n+1/2}\left(\frac{\rho}{2}\sqrt{1+4s}\right)\int_{\rho}^{\infty} K_{n+1/2}\left(\frac{\rho}{2}\sqrt{1+4s}\right)L_n(\rho)\,d\rho$$

Here the functions $L_n(\rho)$ represent the coefficients of expansion of $q^{-1}\rho^{3/2}L(\rho)$ in Legendre polynomials.

After cumbersome calculations we obtain the matching of $\zeta^{*(2)}(\rho, \mu)$ with $\zeta_{*1}(r, \mu)$

$$\zeta^{(2)}(\rho, \mu) = \frac{q}{2s\rho}\left(q\sqrt{1+4s} - \frac{3}{2}\mu\right) - \frac{q}{2s}\ln\rho + \frac{q}{s}Z(s, q, S)\Big) + \tag{3.7}$$

$$\frac{q}{4s}\left(q\sqrt{1+4s} + \frac{3}{4S}\right)\mu - \frac{5q}{24s}\left(1 + \frac{3}{10S}\right)\frac{3\mu^2-1}{2} + O(\rho\ln\rho)$$

$$Z(s, q, S) = -\frac{1+4s}{4}S^2 + \frac{\sqrt{1+4s}}{8}S + \frac{25}{24} - \frac{1+4s}{4}q - \frac{\gamma}{2} -$$

$$\frac{1}{4}\ln(1+4s) + \frac{1}{4}(S\sqrt{1+4s}+1)^2(S\sqrt{1+4s}-2) \times$$

$$\ln\left(1 + \frac{1}{S\sqrt{1+4s}}\right)$$

As is evident from (3.7), for the second approximation of the inner expansion $\alpha_2(P) = P\ln P$. For ζ_2 we have the Laplace equation, the solution of which satisfies the condition (2.3) and the matching condition with (3.7)

$$\zeta_2 = \frac{q}{2s}\left(\frac{q}{r} - 1\right) \tag{3.8}$$

Third approximation. Due to the appearance of a logarithmic singularity in the second approximation of the outer expansion, the third approximation of the inner expansion is also determined by the matching with $\zeta^{(2)}$. We have $\alpha_3(P) = P$, and after all the calculations we obtain

$$\zeta_3 = \frac{q}{s}\sum_{n=0}^{\infty}[\xi_{3,n}(r) + a_n r^n + b_n r^{-n-1}]P_n(\mu) + \tag{3.9}$$

$$q\frac{r}{2} + q^2\frac{\sqrt{1+4s}-1}{4s}\left(1 - \frac{3}{2r} - \frac{1}{4r^3}\right)\mu$$

$$a_0 = Z(s, q, S),\ a_1 = -\frac{\sqrt{1+4s}}{4},\quad a_n = 0,\quad n \geqslant 2$$

$$b_0 = -qZ(s, q, S) - \frac{239}{960} - \frac{79}{240}\frac{1}{k+1} + \frac{1}{32}\frac{1}{(k+1)(k+2)} - \frac{k-1}{k+1}\frac{s}{2}$$

$$b_1 = \left[\frac{7}{16} - \frac{3}{4}\frac{1}{k+2} - \frac{3}{8}\frac{1}{(k+1)(k+2)}\right]\sqrt{1+4s} + \frac{9}{64S}\left(1 + \frac{1}{k+2}\right)$$

$$b_2 = \frac{235}{1344} - \frac{1}{64S}\left(1 + \frac{13}{5}\frac{1}{k+3}\right) + \frac{3}{14}\frac{1}{k+3} + \frac{3}{16}\frac{1}{(k+2)(k+3)}$$

$$b_n = 0,\qquad n \geqslant 3$$

Finally we obtain the third approximation of the outer expansion. If follows from (3.8) and (3.9) that $\alpha^{(3)}(P) = P^2 \ln P$, therefore the equation for $\zeta^{(3)}$ agrees with (3.2). After the matching we obtain

$$\zeta^{(3)} = \frac{q^2}{2s\rho} \exp\left[\frac{\rho}{2}(\mu - \sqrt{1+4s})\right] \tag{3.10}$$

4. Concentration field. Flux of matter on the particle surface. To determine the concentration field near the particle we find the original of the function

$$\zeta_* = \frac{1}{P}\zeta_0 + \zeta_1 + P\ln P\,\zeta_2 + P\zeta_3$$

using the results obtained in Sect. 3. Following the usual method, we obtain for the region $\tau \sim 1$ and $\tau \gg 1$ the concentration field in the form

$$\xi_*(r, \mu, \tau) = P^2 q\Big[\Delta_0(r) + \Delta_1(r)\mu + \Delta_2(r)\frac{3\mu^2 - 1}{2} + \Delta(r, \mu) g(\tau) - \frac{1}{2}\left(\frac{q}{r} - 1\right) G(\tau)\Big] \tag{4.1}$$

Here

$$G(\tau) = \frac{1}{2}\mathrm{Ei}\left(-\frac{1}{4}P\tau\right) - f(\tau) - \frac{3}{2}S\int_0^\tau f'(\tau - x) g(x)\,dx + \frac{1}{2}S^3\frac{d}{d\tau}\int_0^\tau f'(\tau - x)\left[\frac{4}{P}g(x) + \int_0^x g(y)\,dy\right]dx$$

$$\Delta_0(r) = \frac{r}{6} - \frac{\ln r}{2} + \left[\frac{25}{24} - \frac{q}{4} - \frac{\gamma}{2} - \frac{\ln P}{2} - \frac{S^2}{4} - \frac{1}{2}\ln(1 + S^{-1})\right]\left(1 - \frac{q}{r}\right) + \left\{\frac{24}{P^2} + \frac{1}{2-q}\left[\frac{15q - 27}{k+1} + \frac{q(314q - 553)}{40}\right]\right\}\frac{1}{r} + \frac{23 - 16q}{4(2-q)}\frac{1}{r^2} + \frac{1}{2r^3} - \frac{3(3-2q)}{8(2-q)}\frac{1}{r^4} + \frac{1}{10r^5}$$

$$\Delta_1(r) = \frac{1}{2P} + \frac{3}{16S} - \left(\frac{3}{4P} + \frac{9}{32S}\right)\frac{1}{r} + \frac{3-2q}{2-q}\left(\frac{3}{8P} + \frac{9}{64S}\right)\frac{1}{r^2} - \left(\frac{1}{8P} + \frac{3}{64S}\right)\frac{1}{r^3}$$

$$\Delta_2(r) = \frac{1}{12}\Big\{r - \left(\frac{5}{2} + \frac{3}{4S}\right) + \left[\frac{36 - 21q}{4(2-q)} + \frac{9}{8S}\right]\frac{1}{r} - \left[\frac{175 - 110q}{16(2-q)} + \frac{9}{16S}\right]\frac{1}{r^2} + \left(\frac{3}{4} + \frac{9}{20S}\right)\frac{\ln r}{r^3} + \frac{1}{k+3}\left[\frac{73 + 79k}{56} - \frac{84 + 15k}{80S} + \frac{278 - 184q + k(49 - 38q)}{16(2-q)}\right]\frac{1}{r^3} - \frac{3}{8}\left(\frac{3-2q}{2-q} - \frac{1}{S}\right)\frac{1}{r^4} + \frac{5}{56r^5}\Big\}$$

$$\Delta(r, \mu) = \left(\frac{S}{8} - \frac{1}{2P}\right)\left(1 - \frac{q}{r}\right) + \left[-\frac{r}{4} + \frac{q}{4} - \frac{3q}{8r} + \frac{4(k-1) + 3q(k+3)}{16(k+2)}\frac{1}{r^2} - \frac{q}{16r^3}\right]\mu$$

$$g(\tau) = \frac{2}{\sqrt{\pi P\tau}} \exp\left(-\frac{P}{4}\tau\right) + \operatorname{erf}\frac{\sqrt{P\tau}}{2}$$

$$f(\tau) = \frac{1}{2}\operatorname{Ei}\left(-\frac{P}{4}\tau\right) + \frac{1}{\pi}\int_1^\infty \frac{1}{x}\exp\left(-\frac{P}{4}\tau x\right)\operatorname{arctg}(S\sqrt{x-1})\,dx$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)\,du, \quad \operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^u}{u}\,du$$

The calculation of the total flux of matter on the surface of the particle, using (4.1), gives the following relationship for the Nusselt number

$$N(\tau) = 2q + Pq^2 g(\tau) + q^2P^2 \ln P + q^2P^2\left[\frac{q}{2} - \frac{25}{12} + \gamma - \frac{286q - 527}{480(2-q)} + \ln(1 + S^{-1}) - \frac{S}{4}g(\tau) - G(\tau)\right] \qquad (4.2)$$

Thus the passage to the stationary mode is defined by a composite time function. In the region $\tau \to \infty$ the formulas (4.1) and (4.2) yield the results obtained for the stationary mode in [1]. In the particular case of an infinitely large reaction rate

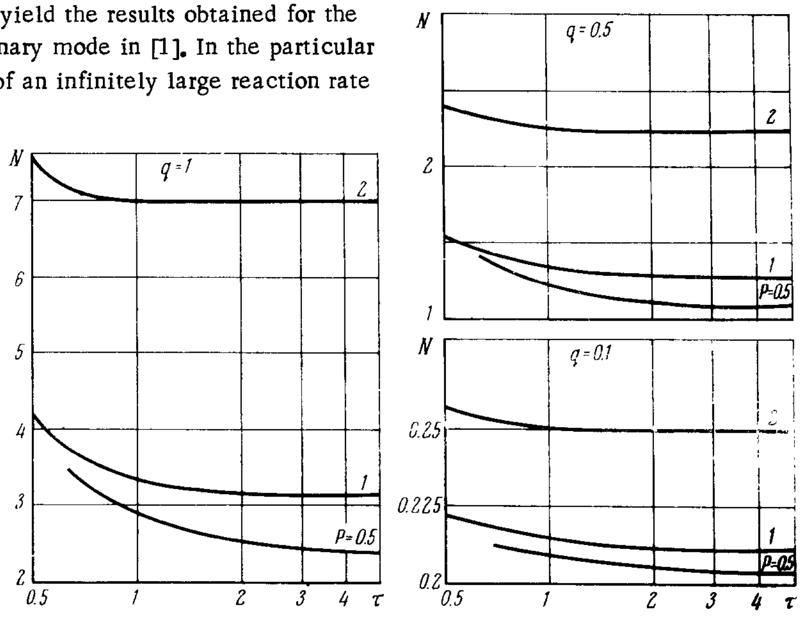

Fig. 1

($q = 1$) they become the relationships obtained in [3].

In Fig. 1 the graphs of the function $N(\tau)$ are given for various values of the Péclet number and various reaction rates. We see that the speed of approach to the stationary mode increases almost inversely proportionally to the Péclet number. The mass transfer process becomes practically the stationary mode for $\tau > 2 / P$, i.e. $t > 2D / U^2$, establishing that the results from [1] are applicable.

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Translated by H. B.

UDC 532.546

FRESH WATER LENS PRODUCED BY UNIFORM INFILTRATION

PMM Vol. 38, №6, 1974, pp. 1048-1055

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(Received February 22, 1974)

The plane model proposed by N. N. Verigin for a stabilized fresh water lens produced by uniform infiltration is investigated in hydrodynamic formulation in the case of equidistant horizontal slit drains. Formulas are obtained for the separation boundary, the depression curve, and characteristic dimesions of the lens.

1. Statement of problem. The considered pattern of flow is shown in Fig. 1. An infinite system of parallel slit drains of the same width $2h$ normal to the xy-plane is disposed along the x-axis (the y-axis is directed vertically upward). We assume that the soil is homogeneous and of unbounded depth, and the distance between the middle of adjacent drains is constant and equal $2L$. Fresh water of density ρ_1 seeps from the surface of the soil over the free boundary (the depression curve AB), passes through the lens (region G), and is drawn off through the drains. Salt ground water of density ρ_2 lies below the separation boundary (curve EF). The case of incomplete flooding of drains is considered and it is assumed that infiltration intensity ε (per unit length of the x-axis) is constant, the ground water is stationary [1, 2], and the motion in the lens is stationary.

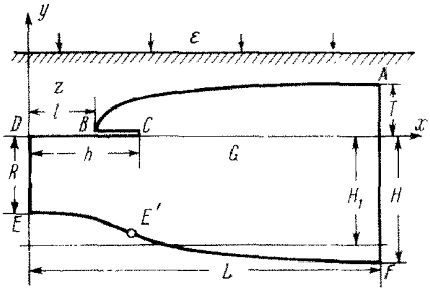

Fig. 1

Investigation of the described model which is periodic with respect to x of period $2L$ and symmetric about the y-axis reduces to the solution of the following mathematical problem [1]. We have to construct region G of the form shown in Fig. 1 with a pair of harmonically conjugate functions φ and ψ inside it, so as to satisfy the boundary conditions